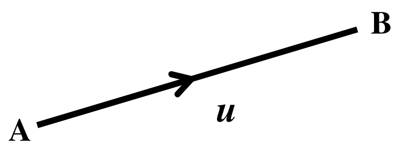
Unit 3 – Chapter 1 Vectors

Scalar: A quantity that has only size, or magnitude. One number can describe it.
Can be measured by a simple scale.
e.g. Temperature, height, mass, distance, speed.

Vector: A quantity that has size (or magnitude) AND direction
– it needs two quantities to describe it.
e.g. displacement (distance moved by a point in a certain direction), force, weight.

Directed line segment:

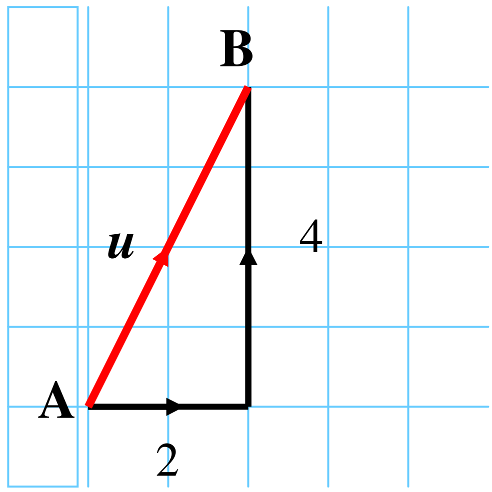
A vector $\boldsymbol{u}$ can be represented in magnitude and direction, by a directed line segment.
The displacement from A to B can be represented by the directed line segment $\overrightarrow{AB}$.



The length of $\overrightarrow{AB}$ is proportional to the magnitude of the vector $\boldsymbol{u}$.
The arrow shows the direction of $\boldsymbol{u}$.

The directed line segment does <u>**not**</u> indicate position,
only magnitude (size) and direction.

Components of a vector in 2 dimensions



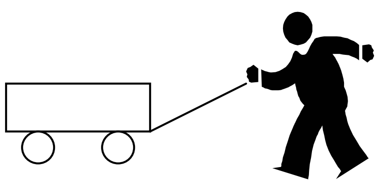
The displacement from A to B can be made by moving
2 units in the x direction and then 4 units in the y direction.

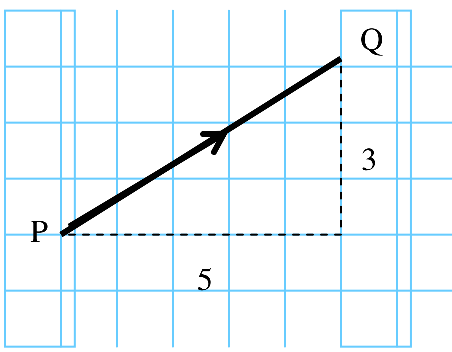
The vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ *or* $\boldsymbol{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

The magnitude of a vector in 2 dimensions

The **length** of the directed line segment represents the **magnitude** (or size) of the vector.

e.g. Jim is pulling a trolley across the floor. He pulls with a force F which has size and direction.
The horizontal component is 5 newtons and the vertical component is 3 newtons.





$$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

The length of PQ is written $\left|\overrightarrow{PQ}\right|$

By Pythagoras Theorem

$$\left|\overrightarrow{PQ}\right|^2 = 5^2 + 3^2 = 34$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{34} = 5.8 \textit{ newtons}$$

This is really the distance formula

Magnitude of a vector $\boldsymbol{u}$

The magnitude of vector $\boldsymbol{u}$ is written $|\boldsymbol{u}|$

If $\boldsymbol{u} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $|\boldsymbol{u}|^2 = (-5)^2 + (7)^2 = 74$ So, $|\boldsymbol{u}| = \sqrt{74}$ *hence*, $|\boldsymbol{u}| = 8.6$

Equal Vectors

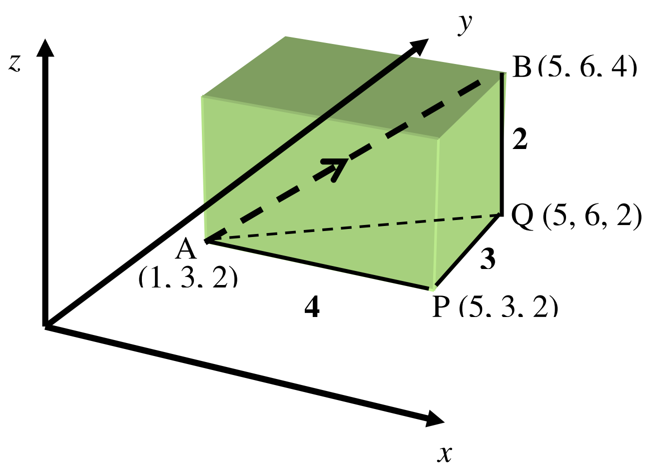
If two vectors are equal, then their components are equal.

e.g. if $\boldsymbol{p} = \begin{pmatrix} a+1 \\ b-1 \end{pmatrix}$ *and* $\boldsymbol{q} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ then a + 1 = 5, so **a = 4** and b – 1 = 4, so **b = 5**

If two vectors have the same magnitude then their lengths will be equal.

e.g. if $\boldsymbol{u} = \begin{pmatrix} x \\ 6 \end{pmatrix}$ *and* $\boldsymbol{v} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ then $x^2 + 6^2 = (-4)^2 + 5^2$ $x^2 = 5$ $x = \pm 2.2$

Vectors in 3 dimensions



The edges of the cuboid are parallel to the axes.

The displacement from A to B can be made by moving:

4 units in the x direction
3 units in the y direction
2 units in the z direction

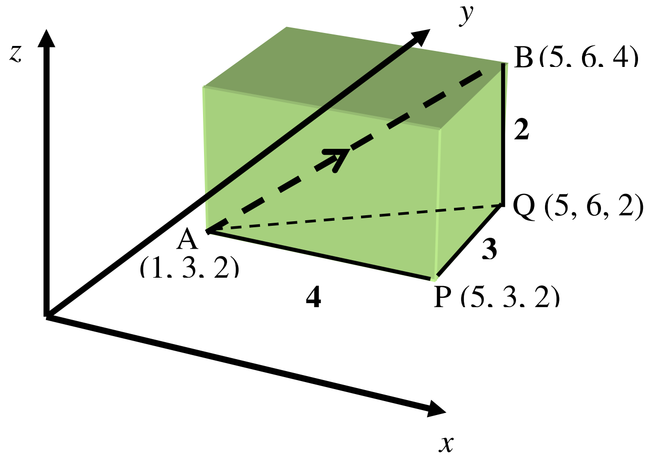
The x component of $\overrightarrow{AB}$ is the step AP, parallel to Ox $= x_B - x_A = 5 - 1 = 4$

The y component of $\overrightarrow{AB}$ is the step PQ, parallel to Oy $= y_B - y_A = 6 - 3 = 3$

The z component of $\overrightarrow{AB}$ is the step QB, parallel to Oz $= z_B - z_A = 4 - 2 = 2$

In component form: $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x\ component \\ y\ component \\ z\ component \end{pmatrix}$ In general $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$, a column vector

Magnitude of a 3 dimensional vector



Using Pythagoras Theorem:

$$AB^2 = AQ^2 + BQ^2$$
$$= \left(AP^2 + PQ^2\right) + BQ^2$$
$$= \left(x_B - x_A\right)^2 + \left(y_B - y_A\right)^2 + \left(z_B - z_A\right)^2$$
$$= 4^2 + 3^2 + 2^2$$
$$= 29$$

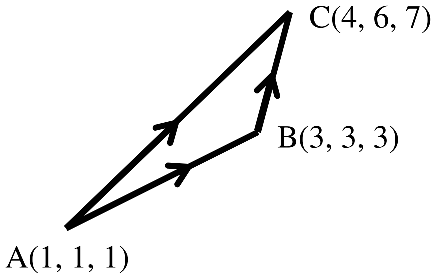
If $\boldsymbol{u} = \overrightarrow{AB}$, the magnitude of $\boldsymbol{u}$, $|\boldsymbol{u}| = AB = \sqrt{29} = 5.4$

$$\boxed{\text{If } \boldsymbol{u} = \overrightarrow{AB}, \text{ then } |\boldsymbol{u}| = AB = \sqrt{\left(x_B - x_A\right)^2 + \left(y_B - y_A\right)^2 + \left(z_B - z_A\right)^2}}$$

This is the Distance Formula for three dimensions.

Addition and subtraction of vectors

An aircraft is spotted at A(1, 1, 1), then at B(3, 3, 3) and then at C(4, 6, 7).



Instead of flying from A to B to C,
it could have flown directly from A to C.

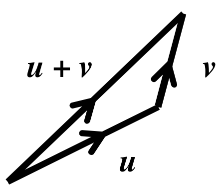
In this sense $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Vectors are, in fact, added together like this – nose to tail.

$$\overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} 4-1 \\ 6-1 \\ 7-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \quad \text{so} \quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Adding two vectors

i) Triangle Rule



ii) Using components $\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} a+p \\ b+q \\ c+r \end{pmatrix}$

The negative of a vector

If $\boldsymbol{u} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$ *and* $\boldsymbol{v} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$, *then* $\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, *the zero vector.*

$\boldsymbol{v}$ is called the negative of $\boldsymbol{u}$, and is written $\boldsymbol{v} = -\boldsymbol{u}$ so $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$

Subtracting two vectors

5 – 3 is the same as 5 + (- 3), subtracting 3 is the same as adding negative 3.

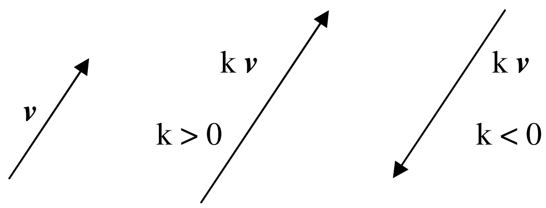
We use the same idea for vectors.

$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$ e.g. If $u = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ *and* $v = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ *then* $u - v = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -8 \end{pmatrix}$

Multiplying by a number (Scalar multiplication)

Given a vector $\boldsymbol{v}$, and a number (scalar) k, then

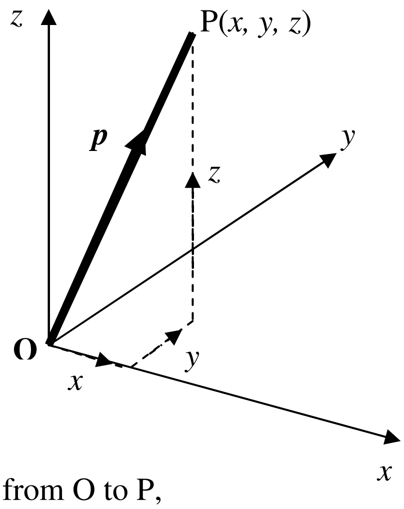
k $\boldsymbol{v}$ has the same direction as $\boldsymbol{v}$ if $k > 0$
k $\boldsymbol{v}$ has the opposite direction as $\boldsymbol{v}$ if $k < 0$



In component form $k \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$

If $\boldsymbol{u} = k\boldsymbol{v} \Leftrightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, $k \neq 0$

Position Vectors



$\overrightarrow{OP}$ is the **position vector** of the point P(x, y, z).

In component form:

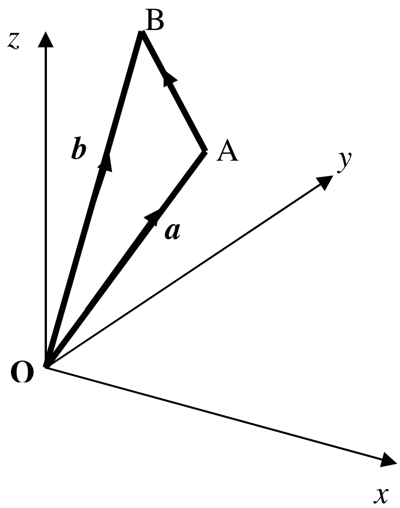
$$\overrightarrow{OP} = \begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad \text{So } \boldsymbol{p} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The co-ordinates (x, y, z), give the position of point P.

While the components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ give instructions for a journey from O to P, with displacements, x, y and z parallel to the x, y and z axes.

The co-ordinates of P are the co-ordinates of its position vector.

A useful result



In the diagram $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

So, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

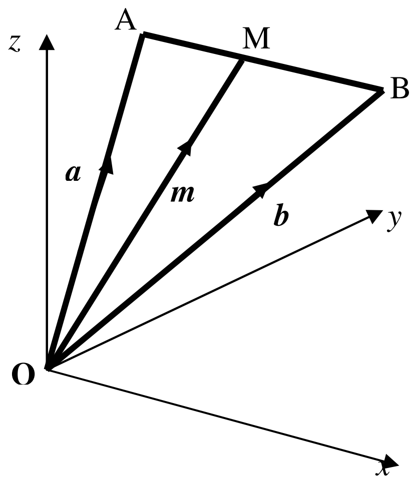
that is $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$\boxed{\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ are the position vectors of A, B.

Similarly $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$, $\overrightarrow{ST} = \boldsymbol{t} - \boldsymbol{s}$ etc.

Position vector of the mid-point of AB



M is the mid-point of AB, and $\boldsymbol{a}$, $\boldsymbol{m}$ and $\boldsymbol{b}$ are the position vectors of A, M and B.

$\overrightarrow{AM} = \overrightarrow{MB}$

so $\boldsymbol{m} - \boldsymbol{a} = \boldsymbol{b} - \boldsymbol{m}$
and $2\boldsymbol{m} = \boldsymbol{a} + \boldsymbol{b}$

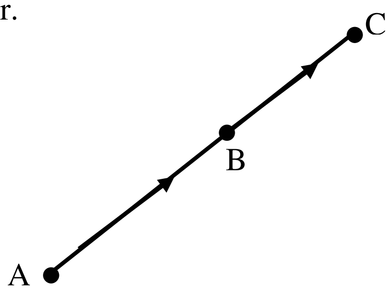
so, $\boldsymbol{m} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$$\boxed{\boldsymbol{m} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})}$$

Collinear Points

Example:

Prove that the points A(2, -3, 4), B(8, 3, 1) and C(12, 7, -1) are collinear.



Find the ratio $\frac{\text{AB}}{\text{BC}}$, i.e. the ratio in which B divides AC

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

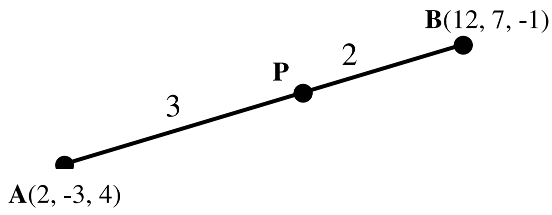
$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} 12 \\ 7 \\ -1 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

So $\frac{1}{3}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC}$ Hence $2\overrightarrow{AB} = 3\overrightarrow{BC}$ so $\overrightarrow{AB} = \frac{3}{2}\overrightarrow{BC}$

Since $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ are scalar multiples, they are parallel, and since B is a common point, they are collinear, and $\frac{\text{AB}}{\text{BC}} = \frac{3}{2}$

Points dividing lines in given ratios

Example: P divides AB in the ratio 3:2
Find the co-ordinates of P



The vector form of $\frac{AP}{PB} = \frac{3}{2}$ is $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{PB}$

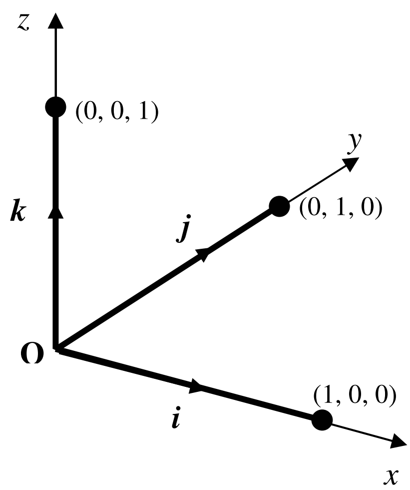
i.e. $2\overrightarrow{AP} = 3\overrightarrow{PB}$

so $2(\boldsymbol{p} - \boldsymbol{a}) = 3(\boldsymbol{b} - \boldsymbol{p})$

$2\boldsymbol{p} - 2\boldsymbol{a} = 3\boldsymbol{b} - 3\boldsymbol{p}$

$5\boldsymbol{p} = 3\boldsymbol{a} + 2\boldsymbol{b}$ Hence $\boldsymbol{p} = \frac{1}{5}\left[\begin{pmatrix}4\\-6\\8\end{pmatrix} + \begin{pmatrix}36\\21\\-3\end{pmatrix}\right] = \begin{pmatrix}8\\3\\1\end{pmatrix}$, so P is (8, 3, 1)

Unit vectors i, j, k



A unit vector has magnitude 1

If $\boldsymbol{u} = \overrightarrow{AB} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is a unit vector then the magnitude of it is 1.

i.e. $a^2 + b^2 + c^2 = 1$.

In particular, unit vectors parallel to the directions OX, OY, OZ are most useful.

We denote these by $\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Every vector can be expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

For example the position vector of the point P(a, b, c) is

$$\boldsymbol{p} = \begin{pmatrix}a\\b\\c\end{pmatrix} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$

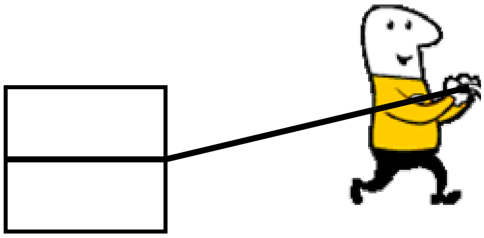
The unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, form a basis for 3-dimensional space.

The Scalar Product of two vectors

We have added and subtracted two vectors and multiplied by a scalar quantity – all of these operations produce another vector.

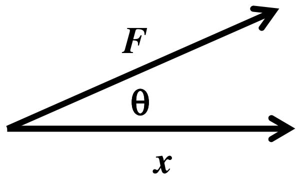
So far there has been no need to multiply two vectors. However, what would be the meaning of two vectors multiplied together and how would we do it ?

Consider a physics analogy -



Tom is pulling the crate across the factory floor with force $\boldsymbol{F}$ (in magnitude and direction).

He moves it through the displacement $\boldsymbol{x}$ (also in magnitude and direction)



Physicists will tell you
the work done is $|\boldsymbol{F}||\boldsymbol{x}|\cos\theta$

In this sense they are multiplying two vectors together even though the product is a real number.

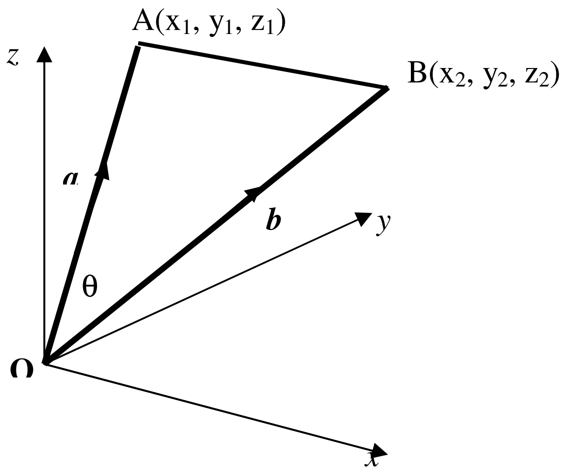
Hence the name **Scalar Product**.

$|\boldsymbol{F}||\boldsymbol{x}|\cos\theta$ is called the scalar product $\boldsymbol{F}.\,\boldsymbol{x}$ of the vectors $\boldsymbol{F}$ and $\boldsymbol{x}$.

Definition: The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a.b}$ ($\boldsymbol{a}$ 'dot' $\boldsymbol{b}$) is

$$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \quad \text{neither } \boldsymbol{a} \text{ nor } \boldsymbol{b} \text{ being zero.}$$

Component form of $\boldsymbol{a.b}$



In ΔOAB,

$AB^2 = OA^2 + OB^2 - 2\,OA\,OB\,\cos\theta$ (cosine rule)

Using the distance formula for OA^2 and OB^2

$$AB^2 = \left(x_1^2 + y_1^2 + z_1^2\right) + \left(x_2^2 + y_2^2 + z_2^2\right) - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta \quad \ldots (1)$$

Now use the distance formula for AB^2

$$AB^2 = \left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2$$

Expanding the brackets and re-arranging:

$$AB^2 = \left(x_1^2 + y_1^2 + z_1^2\right) + \left(x_2^2 + y_2^2 + z_2^2\right) - 2\left(x_1x_2 + y_1y_2 + z_1z_2\right) \quad \ldots (2)$$

From (1) and (2) we see that $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = x_1x_2 + y_1y_2 + z_1z_2$, so, $\boldsymbol{a.b} = x_1x_2 + y_1y_2 + z_1z_2$

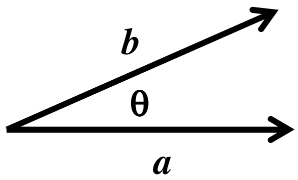
Hence: the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined as:

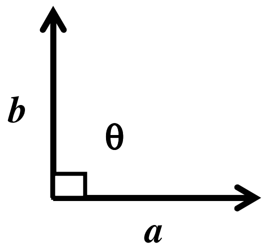
$$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

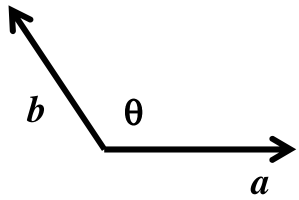
$$\boldsymbol{a.b} = x_1x_2 + y_1y_2 + z_1z_2$$

Sign of the number ***a.b*** (neither ***a*** nor ***b*** being 0)

From its definition, ***a.b*** is a real number whose sign is determined by the size of θ as follows:





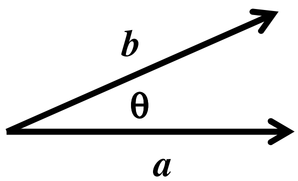


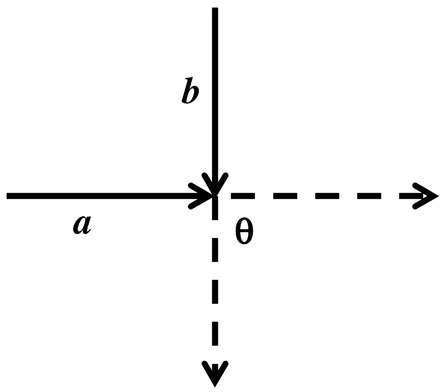
$0° \leq \theta \leq 90°$ $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \; > 0$

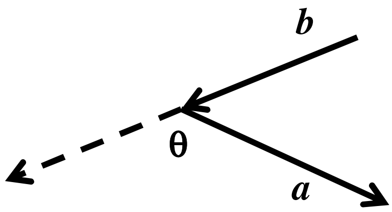
$\theta = 90°$ $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \; = 0$

$90° < \theta \leq 180°$ $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \; < 0$

θ is the angle between vectors **pointing out** from the vertex.







Calculating the angle between two vectors

We can combine the two definitions: $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and $\boldsymbol{a.b} = x_1x_2 + y_1y_2 + z_1z_2$

So that
$$\boxed{\cos\theta = \frac{\boldsymbol{a.b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\boldsymbol{a}||\boldsymbol{b}|} \qquad a \neq 0,\ b \neq 0}$$

Note: $\boldsymbol{a.b} = 0 \iff \theta = 90°$ or $\frac{\pi}{2}$ i.e. ***a*** is perpendicular to ***b*** assuming $a \neq 0,\ b \neq 0$.

Some Results using the Scalar Product

Using $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

(i) $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos 0° = |\boldsymbol{a}||\boldsymbol{a}| \times 1 = a^2$ where a = |***a***| So, $\boxed{\boldsymbol{a}.\boldsymbol{a} = a^2}$

ii) $\boldsymbol{i}^2 = \boldsymbol{i}.\boldsymbol{i} = |\boldsymbol{i}||\boldsymbol{i}|\cos 0° = 1\times1\times1 = 1$ similarly for $\boldsymbol{j}^2$ and $\boldsymbol{k}^2$ So, $\boxed{\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1}$

iii) $\boldsymbol{i}.\boldsymbol{j} = |\boldsymbol{i}||\boldsymbol{j}|\cos 90° = 1\times1\times0 = 0$ similarly for $\boldsymbol{i}.\boldsymbol{k}$ and $\boldsymbol{j}.\boldsymbol{k}$ So, $\boxed{\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{i}.\boldsymbol{k} = \boldsymbol{j}.\boldsymbol{k} = 0}$

The Distributive Law

If $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ *and* $\boldsymbol{c} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

Then $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = x_1(x_2 + x_3) + y_1(y_2 + y_3) + z_1(z_2 + z_3)$

And $\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c} = (x_1x_2 + y_1y_2 + z_1z_2) + (x_1x_3 + y_1y_3 + z_1z_3)$

So it follows that: $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$

A useful result is that if $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = 0$ then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}+\boldsymbol{c}$ are perpendicular.
